MMAT5220 Complex Analysis and its Application

§ 0 Motivation

$\mathbb{N}$: natural numbers

extend $\mathbb{Z}$: integers ($\because$ define subtraction)

extend $\mathbb{Q}$: rational numbers ($\because$ define division)

extend $\mathbb{R}$: real numbers ($\because$ define square root)

extend $\mathbb{C}$: complex numbers ($\because$ define roots of $x^2+1=0$)

extend ?

Let's stay here !

What to do ? Nothing, but extension (generalization)

Extend : algebraic operations $+, -, \times, \div$

function from $\mathbb{C}$ to $\mathbb{C}$

limit, continuity, differentiability, ...

Study the properties, applications !

§ 1 Complex Plane and Elementary Functions

I) Review

$i = \sqrt{-1}$

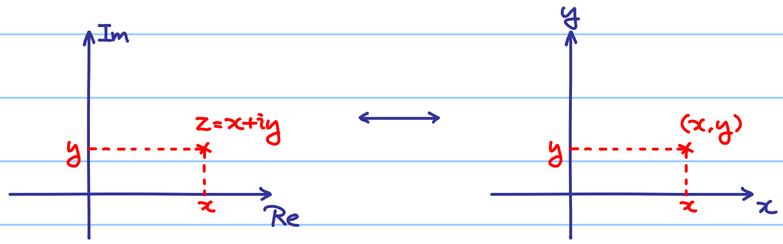
$\mathbb{C}$: set of all complex numbers $\quad \mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$

$\operatorname{Re} z = x$: real part of z

$\operatorname{Im} z = y$: imaginary part of z

one to one correspondence

$$\mathbb{C} \longleftrightarrow \mathbb{R}^2$$



Complex plane

i.e. represent a complex number by a point on a plane.

Regard a real number x as a complex number by writing $x + 0i$

$(\therefore \mathbb{R} \subseteq \mathbb{C})$

Algebraic operations on $\mathbb{C}$

If $z_1 = x_1 + iy_1 \in \mathbb{C}$ and $z_2 = x_2 + iy_2 \in \mathbb{C}$, we define

1) Addition $\quad z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$

2) Subtraction $\quad z_1 - z_2 = (x_1-x_2) + i(y_1-y_2)$

3) Multiplication $\quad z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

4) Division $\quad \dfrac{z_1}{z_2} = \left(\dfrac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + i\left(\dfrac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}\right)$

Remarks :

1) Check : If $z_1, z_2 \in \mathbb{R} \subseteq \mathbb{C}$, $\quad z_1 +_{\mathbb{C}} z_2 \overset{?}{=} z_1 +_{\mathbb{R}} z_2$ and etc.

2) Properties still holds ? $\quad z_1 +_{\mathbb{C}} z_2 \overset{?}{=} z_2 +_{\mathbb{C}} z_1$ and etc.

Ex : Check if $z_1, z_2, z_3 \in \mathbb{C}$

1) Associative law $\quad z_1(z_2z_3) = (z_1z_2)z_3$

2) Commutative law $\quad z_1z_2 = z_2z_1$

3) Distributive law $\quad z_1(z_2+z_3) = z_1z_2 + z_1z_3$

Remark : Conclude all algebraic structures by one statement :

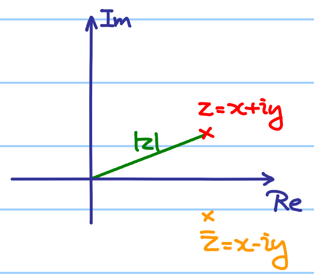
$\mathbb{C}$ is a field ! (Need some arguement)

Modulus :

If $z = x+iy \in \mathbb{C}$, $|z| = \sqrt{x^2+y^2}$ is called the modulus of z

Complex conjugate :

If $z = x+iy \in \mathbb{C}$, $\bar{z} = x-iy \in \mathbb{C}$ is called the complex conjugate of z .



Ex : Check (Geometrical Interpretation ?)

1) $\bar{\bar{z}} = z$

2) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

3) $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

4) $|z| = |\bar{z}|$

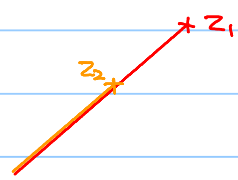
5) $|z|^2 = z\bar{z}$

6) $|z_1z_2| = |z_1||z_2|$

7) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$

8) $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

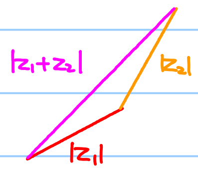
$z_1 \parallel z_2$ if and only if $z_1 = kz_2$ for some $k \in \mathbb{R}$



Triangle inequality :

If $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1+z_2| \le |z_1|+|z_2| ,$$



furthermore , the equality holds if and only if $z_1 \parallel z_2$

proof :

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(\bar{z}_1+\bar{z}_2) \\ &= |z_1|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 + |z_2|^2 \\ &= |z_1|^2 + 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \\ &\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1|+|z_2|)^2 \end{aligned}$$

The equality holds $\Leftrightarrow \mathrm{Re}(z_1\bar{z}_2) = |z_1 z_2|$

$\Leftrightarrow z_1\bar{z}_2 \in \mathbb{R}$

$\Leftrightarrow \frac{z_1}{z_2} \in \mathbb{R}$ Note: $\frac{z_1}{z_2} = z_1\bar{z}_2 \cdot \frac{1}{|z_2|^2}$ $\boxtimes$

Complex polynomial :

$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_i \in \mathbb{C}$.

Fundamental Theorem of Algebra

Every complex polynomial $p(z)$ of degree $n \ge 1$ has a factorization

$p(z) = c(z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$.

(Building block = linear factors Only !)

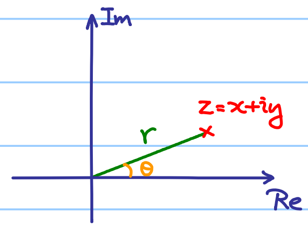
Polar representation :

Cartesian coordinates Polar coordinates

(x, y) (r, θ)

Given r, θ $(r>0)$ Given x, y

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$



(If $x=0, y>0$, $\theta = \frac{\pi}{2}$, NOT defined when $x=y=0$)
$x=0, y<0$, $\theta = -\frac{\pi}{2}$

$z = x + iy = r(\cos\theta + i\sin\theta)$

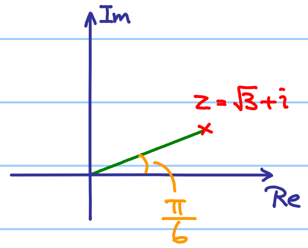
In fact, $r = |z|$

we write $\arg z = \theta$, called argument of z

But θ is determined up to $2m\pi$, $m \in \mathbb{Z}$

$\operatorname{Arg} z$: Principal argument $\quad -\pi < \operatorname{Arg} z \leq \pi$

e.g.



$|z| = 2$

$\arg z = \{ \frac{\pi}{3} + 2m\pi : m \in \mathbb{Z} \}$

$\operatorname{Arg} z = \frac{\pi}{3}$

Remarks :

1) In general, $\arg z = \{ \operatorname{Arg} z + 2m\pi : m \in \mathbb{Z} \}$

2) $\arg z$ is a multivalued function,

i.e. input one complex number, output is a subset of $\mathbb{R}$ instead of a single value

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \in \mathbb{C}$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \in \mathbb{C}$,

Ex: $z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

Hence : $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

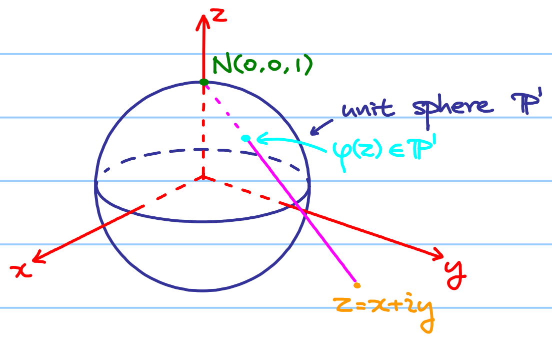
Caution : Does NOT hold for Arg ! (Why ?)

Furthermore : $\arg(\bar{z}) = -\arg(z)$

II) Stereographic Projection

Every complex number can be visualized as a point on a plane.

Another method:



Regard the complex plane as the xy-plane.

Construct a function $\varphi : \mathbb{C} \to \mathbb{P}^1$ by:

1) Given $z = x+iy \in \mathbb{C}$, join z and N by a straight line ℓ

2) ℓ hits the sphere $\mathbb{P}^1$ at a unique point,

then define it to be the image of z under φ

Ex.

In coordinates, $\varphi(x+iy) = \left(\dfrac{2x}{x^2+y^2+1}, \dfrac{2y}{x^2+y^2+1}, \dfrac{x^2+y^2-1}{x^2+y^2+1} \right)$

FACT: 1) φ is injective.

2) $\operatorname{Im} \varphi = \mathbb{P}^1 \setminus \{N\}$.

$\therefore$ Every complex number can be visualized as a point on a sphere.

Interesting properties:

1) If L is a straight line on xy-plane, $\varphi(L)$ is a circle (without N)

Furthermore, if L passes through the origin, $\varphi(L)$ is a great circle passing through N.

2) If $\mathcal{C}$ is a circle on xy-plane, $\varphi(L)$ is a circle

1+2: φ maps straight lines and circles on xy-plane to circles on $\mathbb{P}^1$.

1) suggests N should be intepreted as ∞

If we do so, straight lines on xy-plane should be regarded as generalized circles passing through ∞ with radius $= +\infty$.

(Refer: Non-Euclidean Geometry)

What we need here: Interpretation of ∞!

III) Elementary Functions :

1) Exponential function e^z : (Up to now, we know what e^x is, if $x \in \mathbb{R}$)

$e^z = e^{x+iy} = e^x \cdot e^{iy}$ ← Only thing we have to define !

↑ pretend

Idea: $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$

$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots$

green ⇝ real part

red ⇝ Imaginary part

$$e^{iy} \overset{\text{pretend}}{=} 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= \cos y + i \sin y$$

Putting the above together :

$$e^z = e^{x+iy} \overset{\text{def}}{=} e^x(\cos y + i \sin y)$$

In particular, $e^{i\theta} = \cos\theta + i\sin\theta$ $(x = 0,\ y = \theta)$

(Euler's formula)

Ex : Prove

1) $e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$ (Don't forget, we do NOT have $e^{z+w} = e^z \cdot e^w$ if $z, w \in \mathbb{C}$ up to now !)

2) de Moivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ (But just $(e^{i\theta})^n = e^{in\theta}$)

3) $e^{z+w} = e^z \cdot e^w$

If $z = r(\cos\theta + i\sin\theta) \in \mathbb{C}$, we can write as $z = re^{i\theta}$.

Revisit of complex multiplication :

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

1) $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

2) $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

(More compact form !)

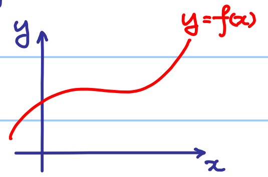
Conclusion : Cartesian coordinates works well for $+, -$

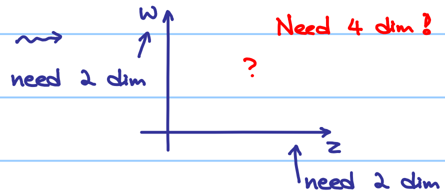
Polar coordinates works well for $\times, \div$

How to visualize the function $w = e^z$?

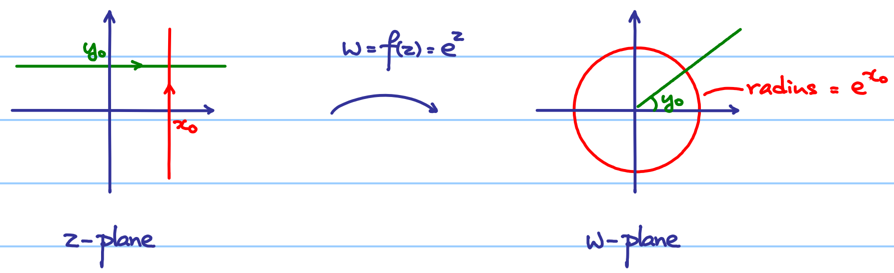
(Real case) (Complex case)

$f : \mathbb{R} \to \mathbb{R}$ $\quad\quad$ $f : \mathbb{C} \to \mathbb{C}$





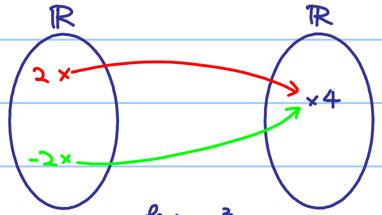
Draw z-plane and w-plane separately!



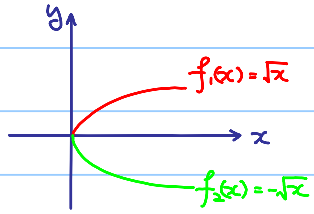
From the above, $\mathrm{Im}\, f = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

2) Square Root

Recall : real case

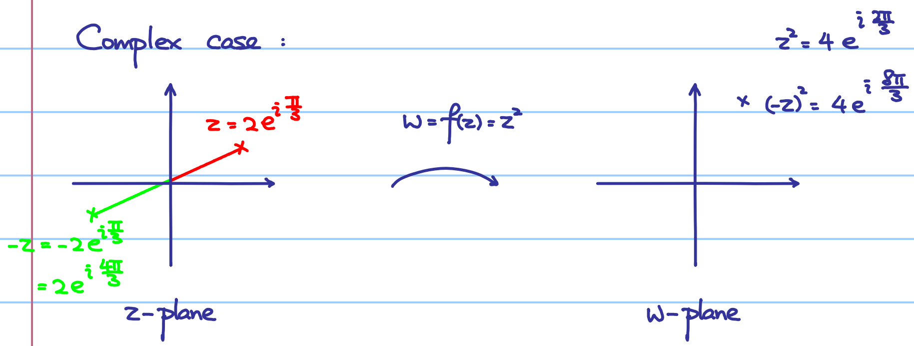


$f(x) = x^2$



To define the inverse, take the red one, ignore the green one.

Complex case :



When we define the inverse (square root), which one we should take ?

No canonical choose !

If $z = re^{i\theta} \in \mathbb{C}^*$

Rough idea : $\sqrt{z} = \sqrt{r}\, e^{\frac{i\theta}{2}}$, but problem comes !

Note : Trouble above comes from $\theta = \arg z$ is a multivalued function

i.e. take $r = 4$, $\theta = \frac{2\pi}{3}$ $\quad \sqrt{z} = 2e^{i\frac{\pi}{3}}$ get the red one

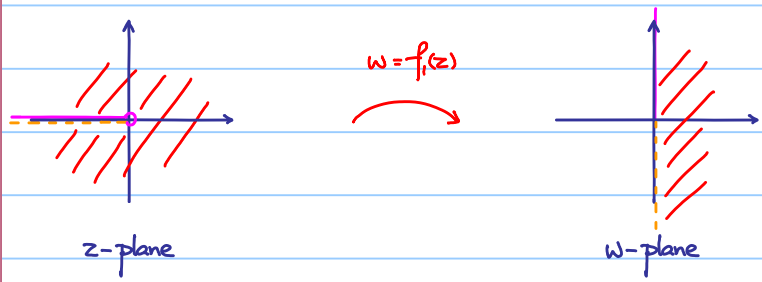
$r = 4$, $\theta = \frac{8\pi}{3}$ $\quad \sqrt{z} = 2e^{i\frac{4\pi}{3}}$ get the green one

Solution : Using $\operatorname{Arg} z$ instead of $\arg z = \theta$

If $z \in \mathbb{C}^*$, we define $f_1(z) = \sqrt{|z|}\, e^{i\frac{\operatorname{Arg} z}{2}}$

$f_2(z) = -\sqrt{|z|}\, e^{i\frac{\operatorname{Arg} z}{2}}$

Recall : $-\pi < \operatorname{Arg} z \le \pi$



f_1 maps $\mathbb{C}^*$ to $\{\operatorname{Re} w \geq 0\} \setminus \{ai \mid a \in \mathbb{R} \text{ and } a \leq 0\}$

Main issue : f_1 is NOT continuous (at points on $(-\infty, 0)$)

$\Downarrow$ Restrict

$f_1 : \mathbb{C}^* \setminus (-\infty, 0) = \mathbb{C} \setminus (-\infty, 0] \longrightarrow \{w : \operatorname{Re} w > 0\}$ Branch cut

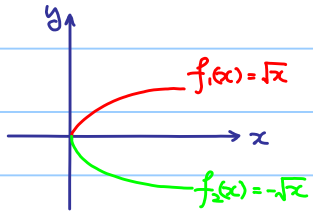
Similar for $f_2 : \mathbb{C} \setminus (-\infty, 0] \longrightarrow \{w : \operatorname{Re} w < 0\}$

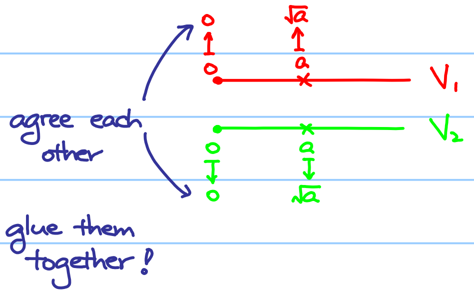
Another point of view :

Real case

$f_1 : [0, +\infty) \to \mathbb{R}$

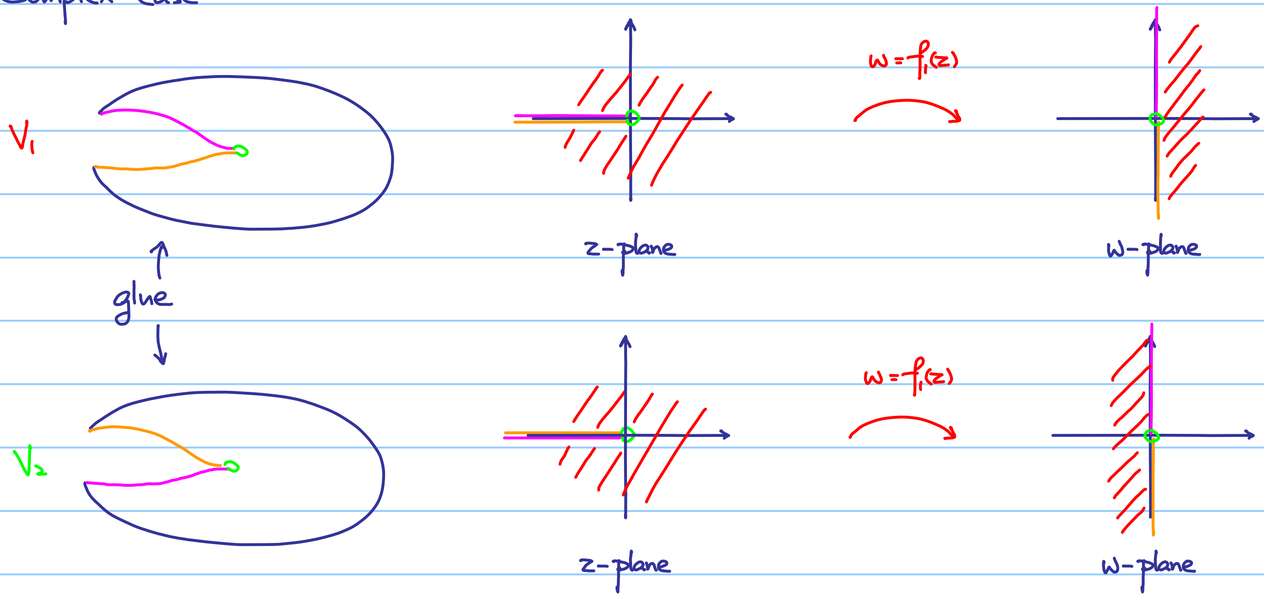
$f_2 : [0, +\infty) \to \mathbb{R}$





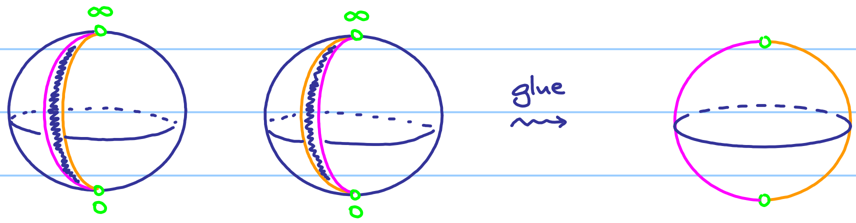
$$\text{Define } f : \underset{\substack{\| \\ V_1 \underset{\{0\}}{\cup} V_2}}{V} \longrightarrow \mathbb{R} \quad \text{by} \quad f(p) = \begin{cases} f_1(p) & \text{if } p \in V_1 \\ f_2(p) & \text{if } p \in V_2 \\ 0 & \text{if } p = 0 \end{cases}$$

Complex case



Similar construction as real case !

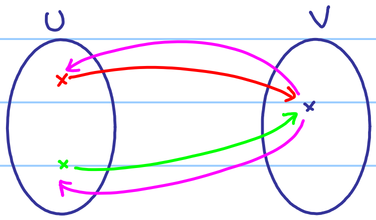
What is the result ?



Remark :

1) Construction of inverse :

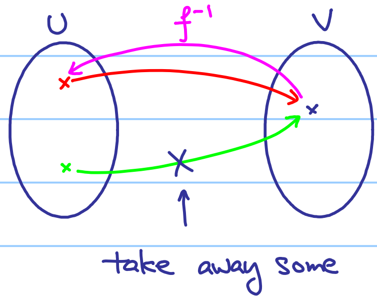
i)



multivalued function (do Nothing)

but NOT a function (can do nothing)

ii)

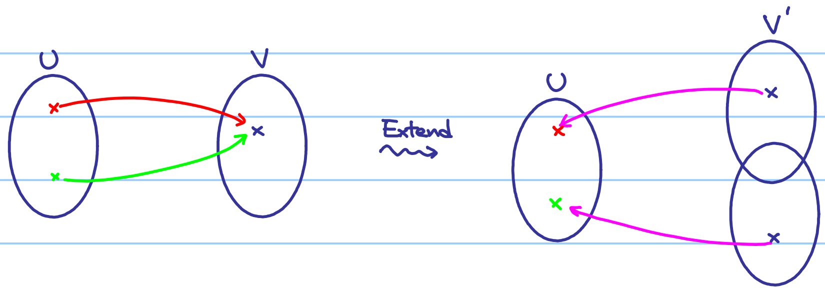


losing information

but easy to do

Trade off !

iii)



No loss of information,

but hard to construct

2) The surface constructed is call a Riemann surface (for $\sqrt{z}$)

Idea : Different Riemann surfaces support different functions.

Look at functions $f : V \to \mathbb{C}$ (OR in general, differential forms, vector bundles, sections)

$\Updownarrow$

Knowing geometry / topology of V.

(Riemannian Geometry)

3) Logarithm Function

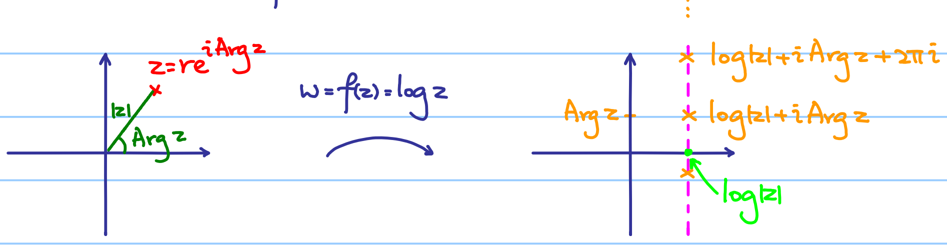
If $z = re^{i\theta} \in \mathbb{C}^*$

$$\underset{\text{in } \mathbb{C}}{\log z} = \log re^{i\theta} \underset{\text{pretend}}{=} \overset{\text{in } \mathbb{R}}{\log r} + \log e^{i\theta} \underset{\text{pretend}}{=} \log r + i\theta$$

It suggests us to define, for $z \in \mathbb{C}^*$,

$$\log z = \log|z| + i\arg z = \log|z| + i\,\mathrm{Arg}\, z + 2m\pi i \quad , \ m \in \mathbb{Z}$$

which is a multivalued function.



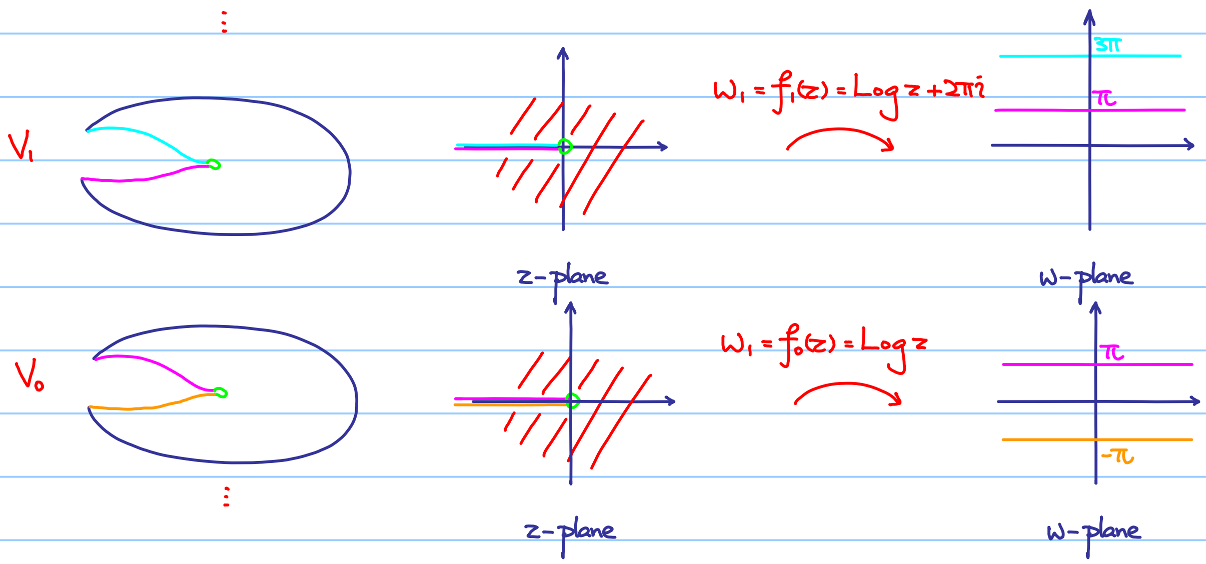
z-plane w-plane

Again, to get a honest function:

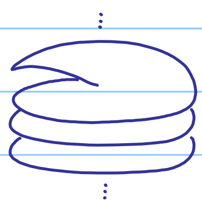
$\mathrm{Log} : \mathbb{C}^* \longrightarrow \{w : -\pi < \mathrm{Im}\, w \leq \pi\}$ defined by $\mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}\, z$

Riemann Surface:

Let $f_m : \mathbb{C}^* \longrightarrow \{w : -\pi + 2m\pi < \mathrm{Im}\, w \leq \pi + 2m\pi\}$ $m \in \mathbb{Z}$



Resulting Riemann surface:



3) Power Function

In real case, $x^\alpha = e^{\log x^\alpha} = e^{\alpha \log x}$, for $x, \alpha \in \mathbb{R}$

It suggests us to define:

If $\alpha \in \mathbb{C}$, $z \in \mathbb{C}^*$, we define

$$\begin{aligned} z^\alpha &= e^{\alpha \log z} \\ &= e^{\alpha[\log|z| + i\operatorname{Arg} z + 2\pi m i]} \qquad m \in \mathbb{Z} \\ &= e^{\alpha \operatorname{Log} z + 2\pi m \alpha i} \end{aligned}$$

as a multivalued function.

But if $\alpha \in \mathbb{Z}$, then $e^{2\pi m \alpha i} = 1$ $\Rightarrow$ the value is NO longer depending on m

$\Rightarrow$ honest function

OR take a branch cut and define

$f_m : \mathbb{C} \setminus [0, -\infty) \to \mathbb{C}$ defined by $f_m(z) = e^{\alpha \operatorname{Log} z + 2\pi m \alpha i}$

e.g.
$$\begin{aligned} i^i &= e^{i \log i} \\ &= e^{i[\log|i| + i\operatorname{Arg} i + 2m\pi i]} \qquad m \in \mathbb{Z} \\ &= e^{i(\log 1 + \frac{\pi}{2} i + 2m\pi i)} \\ &= e^{-(2m + \frac{1}{2})\pi} \end{aligned}$$

Similarly, $i^{-i} = e^{-(2k - \frac{1}{2})\pi}$ $\qquad k \in \mathbb{Z}$

$\therefore \; i^i \cdot i^{-i} \neq i^0 = 1$

Algebraic rules do NOT apply to power functions when they are multivalued.

4) Trigonometric Functions

In real case, $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ for $\theta \in \mathbb{R}$.

$$\Rightarrow \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

It suggests us to define:

If $z \in \mathbb{C}$, we define

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin = \frac{e^{iz}-e^{-iz}}{2i}$$

Exercise: Prove

1) $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$, $z \in \mathbb{C}$

2) $\cos(z+2\pi) = \cos z$ and $\sin(z+2\pi) = \sin z$, $z \in \mathbb{C}$

3) $\cos^2 z + \sin^2 z = 1$, $z \in \mathbb{C}$

4) $\cos(z+w) = \cos z\cos w - \sin z \sin w$, $z, w \in \mathbb{C}$

$\sin(z+w) = \sin z\cos w - \cos z \sin w$, $z, w \in \mathbb{C}$

"proof" of 4)

$$\cos z\cos w - \sin z\sin w = \frac{e^{iz}+e^{-iz}}{2}\cdot\frac{e^{iw}+e^{-iw}}{2} - \frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{e^{iw}-e^{-iw}}{2i}$$

$$= \dots$$

$$= \frac{e^{i(z+w)}+e^{-i(z+w)}}{2} = \cos(z+w)$$

§2 Analytic Functions

I) Limit and Continuity

If $\{s_n\} \subseteq \mathbb{R}$, $\{s_n\}$ converges to $s \in \mathbb{R}$ if

$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $|s_n - s| < \varepsilon \quad \forall\, n \geq N$, denoted by $\lim\limits_{n\to\infty} s_n = s$

Abs. value = dist.

Similarly, if $\{s_n\} \subseteq \mathbb{C}$, $\{s_n\}$ converges to $s \in \mathbb{C}$ if

$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $|s_n - s| < \varepsilon \quad \forall\, n \geq N$, denoted by $\lim\limits_{n\to\infty} s_n = s$

modulus = dist.

Algebraic rules:

If $\{s_n\}, \{t_n\} \subseteq \mathbb{C}$, $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} t_n = t$, then

1) $\lim\limits_{n\to\infty} s_n \pm t_n = s \pm t$

2) $\lim\limits_{n\to\infty} s_n t_n = st$

3) $\lim\limits_{n\to+\infty} \dfrac{s_n}{t_n} = \dfrac{s}{t}$ if $t \neq 0$

If $f : \mathbb{R} \to \mathbb{R}$ is a function, f has a limit $L \in \mathbb{R}$ at $x_0 \in \mathbb{R}$ if

$\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$

denoted by $\lim\limits_{x\to x_0} f(x) = L$.

Similarly, if $f : \mathbb{C} \to \mathbb{C}$ is a function, f has a limit $L \in \mathbb{C}$ at $z_0 \in \mathbb{C}$ if

$\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t. $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

denoted by $\lim\limits_{z\to z_0} f(z) = L$.

Algebraic rules:

If $f, g : \mathbb{C} \to \mathbb{C}$, $\lim\limits_{z\to z_0} f(z) = L$ and $\lim\limits_{z\to z_0} g(z) = M$, then

1) $\lim\limits_{z\to z_0} f(z) \pm g(z) = L \pm M$

2) $\lim\limits_{z\to z_0} f(z) g(z) = LM$

3) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}$ if $M \neq 0$

Let $f:\mathbb{R}\to\mathbb{R}$ ($f:\mathbb{C}\to\mathbb{C}$) be a function,

f is said to be continuous at $x_0\in\mathbb{R}$ ($z_0\in\mathbb{C}$) if

$$\lim_{x\to x_0} f(x) = f(x_0) \qquad \left(\lim_{z\to z_0} f(z) = f(z_0)\right)$$

① limit exists

② f is well-defined at that pt.

③ they equal

Rewrite:

f is said to be continuous at $z_0\in\mathbb{C}$ if

$\forall\, \varepsilon>0$, $\exists\, \delta>0$ s.t. $|f(z)-f(z_0)|<\varepsilon$ whenever $|z-z_0|<\delta$

Let $f:\mathbb{R}^m\to\mathbb{R}^n$ be a function, definition of limit / continuity ?

If $f(x_1,\cdots,x_m) = (f_1(x_1,\cdots,x_m), \cdots, f_n(x_1,\cdots,x_m))$ where $f_i:\mathbb{R}^m\to\mathbb{R}$, $1\le i\le n$

Similarly, if $f:\mathbb{C}\to\mathbb{C}$, $w=f(z)$

$$z=x+iy \mapsto w=u+iv$$

We write $f(z)=f(x+iy) = u(x,y)+iv(x,y)$ by regarding $u,v:\mathbb{R}^2\to\mathbb{R}$

FACT:

If $z_0=x_0+iy_0\in\mathbb{C}$,

1) $\lim_{z\to z_0} f(z) = u_0+iv_0$ if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$

2) $f(z)$ is continuous at z_0 if and only if u and v are continuous at (x_0,y_0)

proof of 1):

" $\Leftarrow$ " Let $\varepsilon>0$, $\exists\, \delta_1,\delta_2>0$ s.t.

$$|u(x,y)-u_0|<\frac{\varepsilon}{2} \quad \text{if} \quad 0<|(x,y)-(x_0,y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_1$$

$$|v(x,y)-v_0|<\frac{\varepsilon}{2} \quad \text{if} \quad 0<|(x,y)-(x_0,y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_2$$

Take $\delta=\min\{\delta_1,\delta_2\}>0$, then if $|z-z_0| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

$$\begin{aligned}|f(z)-(u_0+iv_0)| &= |[u(x,y)-u_0]+i[v(x,y)-v_0]| \\ &\le |u(x,y)-u_0|+|v(x,y)-v_0| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

" $\Rightarrow$ " Exercise

e.g. Let $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$

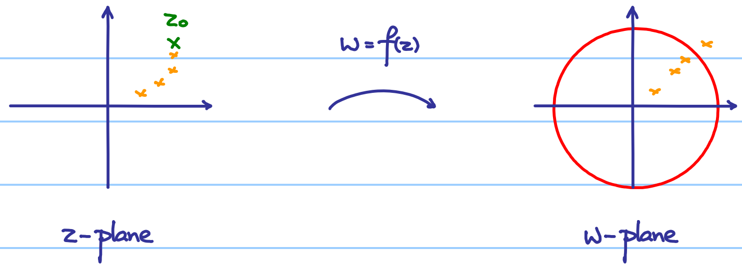
Write $z = x+iy$, $f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy)$ ⇒ $f(z)$ is continuous at every point in $\mathbb{C}$.

continuous everywhere

Limits Involving the Point at ∞.

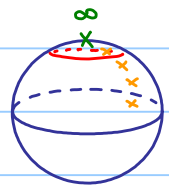
① $\lim\limits_{z \to z_0} f(z) = \infty$?

② $\lim\limits_{z \to \infty} f(z) = w_0$?



Hard to understand converge to ∞ ?

Stereographic projection



① Definition:

$\lim\limits_{z \to z_0} f(z) = \infty$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(z)| > \frac{1}{\varepsilon}$ if $0 < |z - z_0| < \delta$

arbitrary small

arbitrary large

i.e. $\left|\frac{1}{f(z)}\right| < \varepsilon$

$\therefore$ $\lim\limits_{z \to z_0} f(z) = \infty$ if and only if $\lim\limits_{z \to z_0} \frac{1}{f(z)} = 0$

② Definition:

$\lim\limits_{z \to \infty} f(z) = w_0$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ if $|z| > \frac{1}{\delta}$

i.e. $0 < \left|\frac{1}{z}\right| < \delta$

replace by z

i.e. $\left|f\left(\frac{1}{z}\right) - w_0\right| < \varepsilon$ if $0 < |z| < \delta$

$\therefore$ $\lim\limits_{z \to \infty} f(z) = w_0$ if and only if $\lim\limits_{z \to 0} f\left(\frac{1}{z}\right) = w_0$

e.g. Find $\lim\limits_{z\to\infty} \dfrac{2z+i}{z+1}$

$$\lim_{z\to\infty} \frac{2z+i}{z+1} = \lim_{z\to 0} \frac{2\left(\frac{1}{z}\right)+i}{\left(\frac{1}{z}\right)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2$$

Combine ① and ② : Definition :

$\lim\limits_{z\to\infty} f(z) = \infty$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(z)| > \frac{1}{\varepsilon}$ if $|z| > \frac{1}{\delta}$

and $\lim\limits_{z\to\infty} f(z) = \infty$ if and only if $\lim\limits_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.